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GRADIENT-CONSISTENT NON-LINEAR MODEL OF THE GENERATION OF ULTRASOUND IN THE PROPAGATION OF SEISMIC WAVES†

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The equations of the propagation of weak non-linear waves are obtained by a detailed analysis of the gradient-consistent micropolar model of a granular continuous medium. The high-frequency mode of oscillation is associated with ultrasonic waves, and the low-frequency mode is associated with the usual seismic waves in rocks. The evolution equations that include the case of long-wave–short-wave resonance are obtained by an asymptotic consideration. This resonance corresponds to the case of the generation of ultrasound (noise) by travelling seismic waves.

THE PROPAGATION of waves in rocks having a microstructure can be investigated using a well-known generalization of the classical theory of elasticity, namely, the so-called micropolar theory [1]. To explain the experimental fact [2] of the generation of ultrasonic oscillations by ordinary seismic waves or of seismic noise by low-frequency waves the gradient-consistent formulation of the micropolar theory was used in [3]. It essentially employs the condition that the free energy of the medium must depend on the spatial derivatives of the displacement by a unit of greater order than the derivatives of the angle of rotation of a grain. The micropolar model was also used in [4] to analyse non-linear wave processes, but the problem of the generation of high-frequency oscillations by travelling waves was not considered.

1. INITIAL FORMULATION OF THE MATHEMATICAL MODEL

The required variables in the equations of the theory are the coordinates of a particle of the medium x_k and the orthogonal matrix χ_{kl}, describing the rotation of this particle as a solid. These variables are functions of the Lagrange coordinates and time

$$x_k = x_k(X_m,t), \quad \chi_{kl} = \chi_{kl}(X_m,t), \quad k,\ l,\ m = 1,2,3$$

For small strains the displacements u_k and the vector of the angle of rotation φ_k are more convenient; these are connected with x_k and χ_{kl} by the following relations

$$\begin{aligned} &x_k = X_k + u_k \\ &\chi_{kl} = \delta_{kl} + \varphi_{kl} + \tfrac{1}{2}\varphi_{ks}\varphi_{sl} + \ldots \quad (\chi = e^{\varphi}, \varphi_{kl} = -\varepsilon_{klm}\varphi_m) \end{aligned} \tag{1.1}$$

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where ε_{klm} is an alternating tensor, i.e. up to second-order terms

$$\chi_{kl} \approx \delta_{kl} - \varepsilon_{klm}\varphi_m + \tfrac{1}{2}\varphi_k\varphi_l - \tfrac{1}{2}\delta_{kl}\varphi_m\varphi_m \tag{1.2}$$

The equations of motion of a micropolar medium have the form (the balance of momentum and moment of momentum)

$$\begin{gathered}\rho\ddot{u}_k = t_{lk,l}, \quad \rho j\ddot{\varphi}_k = m_{lk,l} + \varepsilon_{klm}t_{lm} \\ (u_{,l} = \partial u / \partial X_l)\end{gathered} \tag{1.3}$$

where t_{lk} and m_{lk} are the stress and moment stress tensors. We will close system (1.3) [3] using the following constitutive relations

$$\begin{aligned} t_{lk} = {} & \frac{\partial\Phi}{\partial G_{KL}} x_{l,K}\chi_{kL} + 2\frac{\partial\Phi}{\partial C_{KLM}} x_{l,K}x_{k,LM} - \\ & - \left[\frac{\partial\Phi}{\partial C_{KLM}}(x_{k,K}x_{l,L}x_{m,M} + x_{l,K}x_{m,L}x_{k,M} - x_{m,K}x_{k,L}x_{l,M})\right]_{,m} \end{aligned} \tag{1.4}$$

$$m_{lk} = \frac{\partial\Phi}{\partial\Gamma_{KLM}}\varepsilon_{RML}x_{l,k}\chi_{kR} \tag{1.5}$$

These equations completely define the volume density of free energy Φ, which depends only on the components of the tensors

$$G_{KL} = x_{s,K}\chi_{sL} - \delta_{KL}, \quad C_{KLM} = x_{s,K}x_{s,LM}, \quad \Gamma_{KLM} = \chi_{sM,K}\chi_{sL} \tag{1.6}$$

The assumption that Φ is a function of only the quantities (1.6) and is the above-mentioned condition of gradient-consistency when the first derivatives of the angle of rotation and the second derivatives of the displacements, respectively, come into play. For small deformations the quantities (1.6) in the principal order are linear in u_k and φ_k. This can be seen from the following expressions for these quantities in terms of u_k and φ_k

$$\begin{aligned} G_{KL} &= u_{L,K} - \varepsilon_{KLm}\varphi_m + \tfrac{1}{2}\varphi_K\varphi_L - \tfrac{1}{2}\delta_{KL}\varphi_m\varphi_m + \varepsilon_{Lsm}u_{s,K}\varphi_m + \ldots \\ C_{KLM} &= u_{K,LM} + u_{s,K}u_{s,LM} + \ldots \\ \Gamma_{KLM} &= \varepsilon_{MLm}\varphi_{m,K} + \tfrac{1}{2}\varphi_L\varphi_{M,K} - \tfrac{1}{2}\varphi_M\varphi_{L,K} + \ldots \end{aligned} \tag{1.7}$$

in which we have taken into account terms of the expansions up to the second power, which will be sufficient later.

When investigating the model of an isotropic micropolar medium in the linear approximation, the free energy density is taken in a uniform quadratic form of the components of the tensors (1.6), invariant under a total orthogonal group. Its general form is derived in [3]. In order to be able to investigate various non-linear phenomena which occur in a medium with a microstructure (such as, for example, the transformation of low-frequency seismic waves into high-frequency waves in the ultrasonic band, connected with the rotational vibrations of the grains or the generation of shear waves by longitudinal waves), we must also take cubic terms into account in Φ. The general form of the invariant uniform cubic form is constructed by a method similar to that described in [3] by exhaustive search for all different methods of convolution in the scalar of the tensor products **GGG**, **GCG**, **GΓΓ** and **GCΓ**, taking into account the symmetry C_{KLM} and the antisymmetry Γ_{KLM} with respect to the indices L and M. Only for these products (the sum of the powers of the trivalent tensors **C** and **Γ** in them are even) does one obtain invariance for reflections. Hence, we will consider a model of the medium which is defined by the following expression for the free energy density

$$\Phi = \Phi_2(\mathbf{G},\mathbf{C},\mathbf{\Gamma}) + \Phi_3(\mathbf{G},\mathbf{C},\mathbf{\Gamma})$$

$$\Phi_2 = a_1 G_{KK} G_{LL} + a_2 G_{KL} G_{KL} + \ldots + a_{13} C_{KLL} \Gamma_{MMK} + a_{14} C_{KLM} \Gamma_{LKM}$$

$$\Phi_3 = a_{15} G_{KK} G_{LL} G_{MM} + a_{16} G_{KL} G_{KL} G_{MM} + \ldots + a_{74} G_{KL} C_{MNN} \Gamma_{MKL} + a_{75} G_{KL} C_{MMN} \Gamma_{NKL}$$

($a_1, \ldots, a_{75}$ are the material constants).

It should be noted that the set of quadratic invariants constructed by the method of all-possible convolutions is complete and linearly independent. At the same time, the system of cubic invariants occurring in Φ_3, being complete, is not linearly independent. There are linear relations between the invariants inside types GCC, $GC\Gamma$ and $G\Gamma\Gamma$. They can be obtained by alternating an eighth-rank tensor, by convolution of which one obtains an invariant, with respect to four indices, arranged in pairs, defining the convolution, for example, from the left, and then carrying out convolution using the same scheme, since alternation over four indices gives an identical zero in three-dimensional space. Hence, the relation between the specific micropolar medium and the set of constants a_i is not one-to-one: different sets of constants may define one and the same medium. In other words, the number of material constants in Φ_3 can be reduced.

However, it is not necessary to carry out this procedure in this paper, since to obtain the non-linear wave effects investigated in the second part of this paper it is sufficient to assume that the combinations of material constants (a small part of them) occurring in the one-dimensional reduction of the general equations are non-zero.

As a result of the closure of the system of equations (1.3) using (1.4) and (1.5), where G_{KL}, C_{KLM}, Γ_{KLM}, x_k, χ_{kl} are expressed in terms of the field variables u_k and φ_k by means of (1.1), (1.2) and (1.7), we obtain the following equations of motion, neglecting terms higher than the second power

$$\rho \ddot{u}_k = a\varepsilon_{k\alpha\beta}\varphi_{\beta,\alpha} + b_1 u_{k,\alpha\alpha} + b_2 u_{\alpha,\alpha k} + b_3 u_{k,\alpha\alpha\beta\beta} + b_4 u_{\alpha,\alpha\beta\beta k} + b_5 \varepsilon_{k\alpha\beta}\varphi_{\alpha,\beta\gamma\gamma} + Q_k^u \tag{1.8}$$

$$\rho j \ddot{\varphi}_k = a\varepsilon_{k\alpha\beta} u_{\beta,\alpha} - 2a\varphi_k + c_1\varphi_{k,\alpha\alpha} + c_2\varphi_{\alpha,\alpha k} + c_3\varepsilon_{k\alpha\beta} u_{\alpha,\beta\gamma\gamma} + Q_k^\varphi \tag{1.9}$$

The coefficients of linear terms can be expressed in terms of the material constants a_i as follows:

$$\begin{aligned} &a = 2(a_1 - a_2), \quad b_1 = 2a_1, \quad b_2 = 2(a_2 + a_3) \\ &b_3 = -a_4 - 2a_6 - 2a_7 + 2a_8, \quad b_4 = -a_4 - 2a_5 - 4a_8 \\ &b_5 = a_{12} + a_{13}, \quad c_1 = 2(a_9 + 2a_{10} + a_{11} + 2a_{12} + a_{14}) \\ &c_2 = -2(a_9 + a_{11} + a_{12}), \quad c_3 = a_{13} - a_{14} + 2a_4 - 4a_8 \end{aligned} \tag{1.10}$$

Comparison with the corresponding formulae from [3] shows certain differences, probably connected with the inaccuracy in the algebraic calculations allowed in [3]. The difference essentially consists of the fact that in [3] the coefficients b_5 and c_3 are identical, whereas according to (1.10) they are independent of one another.

As regards the sets of quadratic terms Q_k^u and Q_k^φ, it is a difficult procedure to write them down and to give explicit expressions for the coefficients in terms of the constants a_i, and is outside the scope of this paper. It is possible best to use computer techniques to obtain them. For our investigation it is sufficient to present some general ideas on the structure of Q_k^u and Q_k^φ which enables us to isolate the non-linear terms on the right-hand sides of Eqs (1.8) and (1.9), which are important for the effects in question when the solution has the form of a plane wave. An analysis of the sets of quadratic terms in (1.8) and (1.9) shows that Q_k^u and Q_k^φ are linear combinations of expressions of the form

$$D_1u_mD_2u_n,\quad D_1u_mD_2\varphi_n,\quad D_1\varphi_mD_2\varphi_n;\quad m,n=1,2,3 \tag{1.11}$$

where the operators D_1 and D_2 are the products of several differentiation operators $\partial/\partial X_\alpha$, where some of the α may be both identical with one another and with m or n.

The order of the operator D_i that acts on u_m can vary from 1 to 4, while the order of the operator acting on φ_n can vary from 0 to 3. As an example, in Q_k^u there are terms

$$cu_{k,\alpha}u_{\alpha,\beta\beta},\quad c'\varepsilon_{\alpha\beta\gamma}u_{\alpha,k}\varphi_{\beta,\gamma},\quad c''\varphi_\alpha\varphi_{k,\alpha\beta\beta} \tag{1.12}$$

etc., where c, c' and c'' are certain coefficients which can be expressed linearly in terms of the material constants a_i (like (1.10)).

The most important characteristic of the quadratic term of the form (1.11) is the order of each of the multiplied derivatives. Corresponding to this, in view of the large number of non-linear terms in (1.8) and (1.9) it is useful to classify the terms in Q_k^u and Q_k^φ according to this criterion, developing them in types and denoting each type by a symbol of the form $u'u''$, $u'u'''$, $u'\varphi''$, etc., where the number of primes is equal to the order of the differential operators D_1 and D_2 in (1.11). The terms (1.12) given above as an example belong to types $u'u''$, $u'\varphi'$ and $\varphi\varphi'''$, respectively.

Although the total number of terms of the form (1.11) in (1.8) and (1.9) exceeds a hundred, the number of different types is much less and can be easily calculated. For this purpose we will formulate a rule which can easily be checked directly by considering in succession all the operations carried out when closing the system of equations (1.3) using (1.4) and (1.5) and taking (1.1), (1.2) and (1.7) into account.

We will introduce the integer l_i which characterizes the operation D_i in (1.11) as follows: l_i is equal to the order of D_i, if D_i acts on φ_m, and l_i is one less than the order of D_i, if D_i acts on u_m. We have already pointed out above the limitation on the order of D_i which we can now express as

$$0\leqslant l_i\leqslant 3,\quad i=1,2 \tag{1.13}$$

The whole variety of types of terms present in Q_k^u and Q_k^φ is determined by the following conditions: (a) $0\leqslant l_1+l_2\leqslant 3$, and (b) for any term from Q_k^u the sum l_1+l_2 is odd, and for any term from Q_k^φ it is even.

Note that condition (b) is a consequence of the symmetry of the model of the micropolar medium being investigated for reflections.

Going by these conditions, i.e. by the fact that in Q_k^u the sum l_1+l_2 is equal to 1 or 3, while in Q_k^φ the sum l_1+l_2 is equal to 0 or 2, and taking (1.3) into account, we can indicate all types of quadratic terms in (1.8) and (1.9), namely

the types of terms in Q_k^u

$$\begin{gathered}u'u'',\ u'u'''',\ u''u''';\ u'\varphi',\ u'\varphi''',\ u''\varphi,\ u''\varphi''\\ u'''\varphi',\ u''''\varphi;\ \varphi\varphi',\ \varphi\varphi''',\ \varphi'\varphi''\end{gathered} \tag{1.14}$$

and the types of terms in Q_k^φ are

$$u'u',\ u'u''',\ u''u'';\ u'\varphi,\ u'\varphi'',\ u''\varphi',\ u'''\varphi;\ \varphi\varphi,\ \varphi\varphi'',\ \varphi'\varphi' \tag{1.15}$$

It should be noted that there are, in fact, no terms of the type $\varphi\varphi$ since they can only have the form $\varepsilon_{k\alpha\beta}\varphi_\alpha\varphi_\beta$ or $\varepsilon_{\alpha\beta\beta}\varphi_\alpha\varphi_k$, which is obviously equal to zero.

It can also be seen that the linear terms are subject to rules similar to (a) and (b), only simpler. They are characterized by a single operator D, for which l satisfies the conditions: $0\leqslant l\leqslant 3$; l is odd in the first of equations (1.8) and (1.9) and even in the second of these.

Note that all possible terms in Q_k^u and Q_k^φ with arbitrary coefficients c_j can be calculated by forming tensor products from the partial derivatives of u_k and φ_k belonging to the above-mentioned types

$$u_{\alpha,\beta}u_{p,qr},\quad u_{\alpha,\beta}u_{p,qrst},\ldots;\quad \varepsilon_{\lambda\mu\nu}u_{\alpha,\beta}\varphi_{p,q}$$

$$\varepsilon_{\lambda\mu\nu}u_{\alpha,\beta}\varphi_{p,qrs},\ldots;\ \ldots\ \ldots;\ \ldots,\quad \varepsilon_{\lambda\mu\nu}\varphi_{\alpha,\beta}\varphi_{p,q} \tag{1.16}$$

($\varepsilon_{\lambda\mu\nu}$ is supplemented with a calculation such that the overall number of indices is odd), and by inspecting all non-equivalent methods of allocating one of the indices of value k and convoluting with respect to the remaining, pairwise identical, indices. The identically zero terms, like the above type $\varphi\varphi$, are naturally ignored. It is still a difficult problem to determine how the coefficients c_j depend on the constants a_i, which we mentioned above. We cannot simply calculate the different coefficients for all possible terms of the type (1.16) (i.e. assume these coefficients to be independent), since even in the linear parts of Eqs. (1.8) and (1.9) a relationship between them is observed: the coefficients of $\varepsilon_{k\alpha\beta}\varphi_{\beta,\alpha}$, $\varepsilon_{k\alpha\beta}u_{\beta,\alpha}$ and φ_k are proportional. Moreover, the presence of such relationships between the coefficients of the quadratic terms is unavoidable, since the number of different forms of these terms obtained by the method described above exceeds the number of constants a_i.

2. NON-LINEAR LONGITUDINAL WAVES IN A MICROPOLAR MEDIUM

We will consider one-dimensional non-linear waves in a micropolar medium described by the general equations. For a wave propagating in the direction n

$$u_i(X_\alpha,t)=u_i(X,t),\quad \varphi_i(X_\alpha,t)=\varphi_i(X,t)$$
$$X=X_\alpha n_\alpha,\quad \alpha,i=1,2,3 \tag{2.1}$$

The spatial derivatives here are converted by the following rule

$$f_{i,\alpha}=n_\alpha\partial f/\partial X\equiv n_\alpha f' \tag{2.2}$$

We will direct the X axis along the direction of propagation of the wave, i.e. $\mathbf{n}=(1,0,0)$.

The non-linear equations (1.8) and (1.9) contain both longitudinal and transverse components of u_α and φ_α, and hence the separation of the non-linear waves into longitudinal and shear waves is fairly conventional. We will consider, however, the non-linear waves which are converted into longitudinal waves in the linear limit [5]. These are the equations for $\varphi\equiv\varphi_1$ and $u\equiv u_1$. For brevity we will call the waves investigated longitudinal waves. The equations for u and φ take the form

$$\rho\ddot{u}=2(a_1+a_2+a_3)u''-2(a_4+a_5+a_6+a_7+a_8)u''''+Q^u \tag{2.3}$$

$$\rho j\ddot{\varphi}=2(2a_{10}+a_{14})\varphi''-4(a_1-a_2)\varphi+Q^\varphi \tag{2.4}$$

Note that the linear terms of the type u''' which occur in the general equation (1.9) for microrotations, do not occur in Eq. (2.4) for the longitudinal wave.

It can be seen that in the linear approximation ($Q^u=0$, $Q^\varphi=0$, $\rho=\rho_0$) Eqs. (2.3) and (2.4) are uncoupled.

For convenience and in order to agree with [5], we will introduce the following quantities

$$c_1^2=2(a_1+a_2+a_3)/\rho_0,\quad c_2^2=2(2a_{10}+a_{14})/\rho_0 j$$
$$\delta=-2(a_4+a_5+a_6+a_7+a_8)/\rho_0,\quad \omega_0^2=4(a_1-a_2)/\rho_0 j \tag{2.5}$$

We will consider the problem of the evolution of a longitudinal seismic wave. Suppose that when $t=0$

$$u=u_1,\quad u_2=u_3=0,\quad \varphi_1=\varphi_2=\varphi_3=0 \tag{2.6}$$

The transverse components of the vector $\mathbf{u}$ only appear because of the non-linear transformation (the presence of a longitudinal component in Q^u—of non-linear terms in the equations for u_2 and u_3) and can therefore be regarded as small compared with $u\equiv u_1$ and we can consider the mass balance in the form

$$\rho_0/\rho=1+u' \tag{2.7}$$

High-frequency longitudinal waves of microrotation may manifest themselves in this system because of the non-linear mechanism by which energy is transferred from longitudinal micromotions to microrotations—so-called long-wave–short-wave resonance [6]. Here, obviously, in problem (2.6) $\varphi_1 \gg \varphi_2, \varphi_3$, since the transverse components φ_2 and φ_3 can only manifest themselves as a result of long-wave–short-wave resonance with small transverse components u_2 and u_3, and also due to non-linear transformation of the component φ_1 itself.

The non-linear equations for the longitudinal waves therefore take the form

$$\ddot{u}-c_1^2u''-\delta u''''-\nu u'u''+2\chi\varphi'\varphi+Q_*^u=0 \tag{2.8}$$

$$\ddot{\varphi}-c_2^2\varphi''+\omega_0^2\varphi-u'(c_2^2\varphi''-\mu\varphi)+Q_*^\varphi=0 \tag{2.9}$$

The coefficients ν and χ can easily be calculated by grouping terms of the type $u'u''$ and $\varphi\varphi'$ in Q^u (see (1.14)). The sum of the remaining non-linear terms Q_*^u, as shown below, makes no contribution to the effects being investigated. Similarly, by grouping terms in Q^φ proportional to $u'\varphi$ (see (1.15)), we can calculate the coefficient μ in (2.9) (the term Q_*^φ also turns out to be unimportant in the problem in question).

System (2.8), (2.9) has a clear structure.

Equation (2.8) is an integrable Boussinesq equation, supplemented by a non-linear term which describes the interaction between the high-frequency and low-frequency oscillations. The term with the leading derivative corresponds to dispersion of the long waves. This equation can be reduced, by means of a standard procedure of extending the coordinates and expanding the dependent variable in a small parameter (see for example, [6]) to the well-known single-wave Korteweg–de Vries equation, which describes the long weakly dispersing displacement waves quite well, but does not take into account the effect of high-frequency oscillations of the microvibrators. Equation (2.9) is a linear Klein–Gordon equation, supplemented by bilinear components characterizing the effect of low-frequency (seismic) oscillations on the high-frequency (ultrasonic) waves. As we know (see, for example, [6]), the integrable model to which the Klein–Gordon equation leads (taking into account the non-linear terms from Q^φ), is a non-linear Schrödinger equation for the envelope of the high-frequency wave. This equation, like the Korteweg–de Vries equation, does not take into account the interaction between waves of different scales, which is of fundamental interest to us. This interaction can be described using the long-wave–short-wave resonance model.

3. RESONANCE OF LONG AND SHORT WAVES

The linear waves of system (2.8), (2.9) are characterized by the dispersion relations

$$\omega_s^2=c_1^2k^2+\delta k^4 \tag{3.1}$$

for the seismic waves (subscript s) and

$$\omega_{us}^2 = c_2^2 k^2 + \omega_0^2 \tag{3.2}$$

for the ultrasonic waves (subscript us) (see Fig. 1).

Suppose the ratio $\varepsilon = k_s / k_{us}$ of the characteristic wave numbers of the seismic and ultrasonic waves is small. We will represent the required variables u and φ in the form of asymptotic series in the parameter ε, using the well-known technique of multiscale expansions [6]

$$u(\xi,\tau) = \varepsilon u^{(1)} + \ldots, \quad \varphi(x,t,\xi,\tau) = \varepsilon^q \varphi^{(1)} + \varepsilon^{q+1}\varphi^{(2)} + \ldots \tag{3.3}$$

where $\xi = \varepsilon(x - c_g t)$, $\tau = \varepsilon^2 t$ are "slow" coordinates, and $c_g = d\omega_{us}/dk$ is the group velocity of the ultrasonic waves, and $q > 0$. The derivatives $\partial/\partial t$ and $\partial/\partial x$ when introducing the new set of independent variables $x, t; \xi, \tau$ are converted as follows:

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \varepsilon c_g \frac{\partial}{\partial \xi} + \varepsilon^2 \frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial \xi} \tag{3.4}$$

Substituting the expansions (3.3) and (3.4) into Eq. (2.4) of the dynamics of the microrotation, and equating the coefficients of like powers of ε to zero, we obtain a system of coupled equations. The first of these corresponds to the factor ε^q and leads to separation of the fast and slow variables

$$\varphi^{(1)} = A(\xi,\tau)e^{i\theta} + \text{c.c.}, \quad \theta = kx - \omega t + \theta_0 \tag{3.5}$$

where θ_0 is an arbitrary constant and A is a slowly varying complex amplitude (envelope). The required equation for A follows from the equation for $\varphi^{(3)}$ corresponding to the order $q+2$

$$\ddot{\varphi}^{(3)} - c_2^2 \varphi^{(3)\prime\prime} + \omega_0^2 \varphi^{(3)} = [(c_2^2 - c_g^2)\partial^2 A / \partial\xi^2 + 2ik\partial A / \partial\tau -$$
$$- VA(\omega_0^2 + k^2 c_2^2)]e^{i\theta} + \text{c.c.} + Q_*^{\varphi}, \quad V = \partial u^{(1)} / \partial\xi \tag{3.6}$$

The quantity in square brackets gives rise to a secular increase in the approximation considered. The condition for this increase not to occur leads to the equation of the envelopes

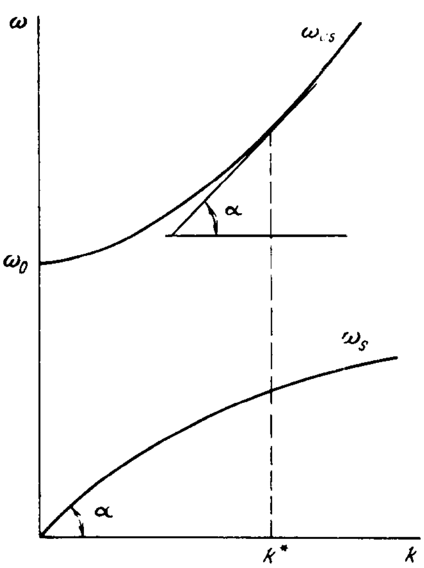

FIG. 1.

$$2ikA_\tau + (c_2^2 - c_g^2)A_{\xi\xi} = (k^2c_2^2 + \omega_0^2)VA \tag{3.7}$$

It can be seen that the non-linear terms in Q_*^φ (their structure is described in (1.15)), occur in the higher approximation in ε, and make no contribution to the secular increase in $\varphi^{(3)}$. It should be noted that the quadratic terms in φ and Q_*^φ nevertheless lead to terms of the type $|A|^2A$, characterizing the self-action of the ultrasonic waves on the right-hand side of Eq. (3.7). Taking them into account later does not lead to any fundamental changes. Moreover, when investigating the interaction between waves corresponding to different branches of the dispersion relation, it is natural to assume that these terms are small compared with the bilinear term $\sim VA$ considered.

To obtain an equation for the deformation to a first approximation $V(\xi, \tau)$ we must also change to coordinates ξ, τ instead of x, t in (2.8)

$$(c_g^2 - c_1^2)V_{\xi\xi} - 2\varepsilon c_g V_{\tau\xi} = 2\chi\varepsilon^{2(q-1)}|A|_{\xi\xi}^2 - (Q_*^u)_\xi / \varepsilon^2 \tag{3.8}$$

It turns out that the non-linear term $\nu u'u''$ and the non-local term $\delta u''''$ in the new coordinates are small, and they can be neglected in (3.8). In addition, when using (1.14) it can be shown that there are no terms in Q_*^φ which might compete with those taken into account, and hence the term Q_*^u can also henceforth be neglected.

When there is no resonance ($c_g \neq c_1$) the second term on the left-hand side of (3.8) turns out to be much less than the first, so that by choosing $q = 1$ we obtain

$$V = -2\chi(c_1^2 - c_g^2)^{-1}|A|^2 \tag{3.9}$$

Substituting (3.9) into (3.7) we obtain a non-linear Schrödinger equation for the envelope of the fast field

$$2ikA_\tau + (c_2^2 - c_g^2)A_{\xi\xi} = -2\chi(k^2c_2^2 + \omega_0^2)(c_1^2 - c_g^2)^{-1}|A|^2 A \tag{3.10}$$

Thus, in the non-resonance situation considered the envelope of the ultrasonic wave and the slow (seismic) variable are rigidly connected by the algebraic relation (3.9), i.e. the lower branch "copies" the upper branch.

In the case of resonance there is a certain value k^* for which the following equation is satisfied

$$c_g(k^*) = c_1 \tag{3.11}$$

i.e. the group velocity of the ultrasonic waves c_g is identical with the phase velocity of long seismic waves: $c_s = \omega_s / k \to c_1(k \to 0)$ (see Fig. 1). We will assume that the interaction constant χ is of order of smallness ε^p, i.e.

$$\chi = \varepsilon^p c_1 \bar{\chi}, \quad p > 0, \quad \bar{\chi} = O(1) \tag{3.12}$$

Then, choosing $q = (3 - p)/2$, Eq. (3.8) takes the form

$$V_\tau = \bar{\chi}|A|_\xi^2 \tag{3.13}$$

The value of p itself in (3.12) is determined by the actual value of the small elastic coupling constant χ.

Introducing the new variables

$$L = V\frac{(k^*)^2c_2^2 + \omega_0^2}{2k^*}, \qquad S = A\left[\frac{|\bar{\chi}|(\omega_0^2 + (k^*)^2c_2^2)}{((c_2^2 - c_1^2)2k^*)^{1/2}}\right]^{1/2}$$

$$t' = \tau, \quad x' = \xi(2k^*)^{1/2}(c_2^2 - c_1^2)^{-1/2} \tag{3.14}$$

we obtain a canonical long-wave–short-wave resonance system (see, for example, [6])

$$2iS_t + S_{xx} = 2LS, \quad L_t = \pm|S|_x^2 \tag{3.15}$$

The sign in the second equation is governed by the sign of the constant $\overline{\chi}$, and the primes on the independent variables are omitted. System (3.15) arises in different resonance situations, for example, when investigating the interaction between Langmuir and ionosonic waves in a plasma [7], and also when describing the protein α-helix [8]. Soliton solutions of the long-wave–short-wave resonance equations have been investigated in [9, 10].

To fix our ideas we will consider system (3.15) with the "minus" sign in the second equation. By making the replacement of variables

$$S = I^{1/2}e^{i\varphi}, \quad \varphi_x = w \tag{3.16}$$

these equations can be reduced to a hydrodynamic-type system

$$\begin{aligned} &I_t + (Iw)_x = 0, \quad L_t + I_x = 0 \\ &w_t + [w^2/2 + L - \tfrac{1}{4}(I_{xx}/I - I_x^2/2I^2)]_x = 0 \end{aligned} \tag{3.17}$$

The so-called dispersionless limit, which occurs for smooth "flows", i.e. when $I/I_x \gg 1$, is of interest. The third of Eqs. (3.7) then takes the simpler form

$$w_t + (w^2/2 + L)_x = 0 \tag{3.18}$$

and system (3.17), (3.18) can be represented in the Riemann form

$$\partial r_i/\partial t + V_i(\mathbf{r})\partial r_i/\partial x = 0, \quad i = 1,2,3 \tag{3.19}$$

where r_i are Riemann invariants, and $V_i(r)$ are characteristic velocities which depend on all three invariants, and there is no summation over repeated indices.

The connection between the Riemann invariants and the characteristic velocities with initial variables I, w, and L is given by the relations

$$r_i = L + w^2(\tfrac{3}{2}\Lambda_i - 1)\Lambda_i, \quad V_i = w\Lambda_i \tag{3.20}$$

where Λ_i are the roots of the cubic equation

$$\Lambda(\Lambda - 1)^2 = I/w^3 \tag{3.21}$$

System (3.17), (3.18) and, correspondingly, (3.19) is hyperbolic, if

$$0 < I/w^3 < 4/27 \tag{3.22}$$

In this case the envelope of the generated ultrasonic wave is stable, and to describe the smooth evolution of the required variables it is sufficient to use system (3.19) instead of the considerably more complex system (3.15). Thus, any $r_i = \text{const}$ are exact solutions of (3.19), which also enables its analysis to be simplified considerably.

If condition (3.22) is not satisfied, the "smooth" regime breaks up rapidly due to modulation instability and the leading derivatives play a role in the second equation of system (3.17). The envelope of the ultrasonic packet itself then begins to oscillate, which leads to decay into a soliton (envelope).

Finally, using the resonance condition (3.11) the wavelength λ^* of the excited ultrasonic wave can be expressed in terms of the material constants as follows:

$$\lambda^* = 2\pi c_2 (c_2^2 - c_1^2)^{1/2} / (c_1 \omega_0) \tag{3.23}$$

In other words λ^* is independent of the frequency of the seismic vibrations and is proportional to the internal scale of the medium. The long-wave–short-wave resonance approximation will then work more accurately the more rigorously the following inequality is satisfied

$$\lambda^* / \lambda_s \sim \varepsilon \ll 1$$

where λ_s is the wavelength of the action, and

$$\lambda_s \gg \delta^{1/2} / c_1$$

The last condition follows from the linear dispersion relation (3.1) and denotes that the most effective resonance is obtained in the linear-dispersion region.

For the replacement (3.14) to be correct it is necessary for the inequality $c_2 > c_1$ to be satisfied. The ratio of the phase velocities of the excited ultrasound and of the seismic wave then satisfies the relation

$$c_{us} / c_s = c_2 / c_1 > 1$$

which agrees with experiment [2].

REFERENCES

1. ERINGEN A. K., The theory of micropolar elasticity. *Fracture*, Vol. 2, Mir, Moscow, 1975.
2. VILCHINSKAYA N. A. and NIKOLAYEVSKII V. N., Accoustic emission and the spectrum of seismic signals. *Izv. Akad. Nauk. Fiz. Zemli* 5, 91–100, 1984.
3. BRULIN O. and HJALMARS S., Linear grade-consistent micropolar theory. *Int. J. Engng Sci.* **19**, 12, 1731–1738, 1981.
4. ERBAY S. and SUHUBI E. S., Nonlinear wave propagation in micropolar media. I, II. *Int. J. Engng Sci.* **27**, 8, 895–919, 1989.
5. KRYLOV A. L., NIKOLAYEVSKII V. N. and EL' G. A., A mathematical model of the non-linear generation of ultrasound by seismic waves. *Dokl. Akad. Nauk SSSR* **318**, 6, 1340–1344, 1991.
6. DODD R. K. *et al.*, *Solitons and Nonlinear Wave Equations*. Academic Press, London, 1982.
7. ZAKHAROV V. A., The collapse of Langmuir waves. *Zh. Eksp. Teor. Fiz.* **62**, 5, 1745–1759, 1972.
8. DAVYDOV A. S., *Solitons in Molecular Systems*. Nauk. Dumka, Kiev, 1988.
9. MA Y. C., The complete solution of the long-wave–short-wave resonance equations. *Stud. Appl. Math.* **59**, 3, 201–221, 1978.
10. YAJIMA N. and OIKAWA M., Formation and interaction of sonic-Langmuir solitons: Inverse scattering methods. *Progr. Theor. Phys.* **56**, 6, 1719–1739, 1976.

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